CHAPTER THREE

3.0 AREAS AND VOLUMES

3.1 AREAS DETERMINATION

One of the major tasks of cadastral and engineering survey is the determination of the sizes of the plots involved.

Such areas fall into one of three categories, which area

(i) Straight sided
(ii) Irregular sided or
(iii) A combination of both

3.1 Areas enclosed by straight lines.

The results obtained for such areas will be exact since correct geometric equations and theorems can be applied

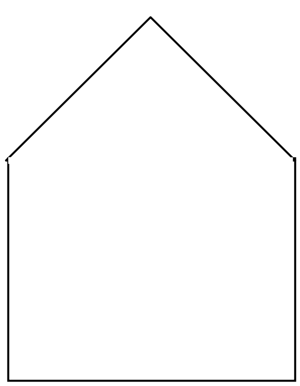
3.1.1 Areas from triangles

A straight-sided figure can be divided into well-conditioned triangles, the areas of which can be calculated using of the following formula

(1) Area = $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the sides of the triangle and s= ½(a+b+c)
(2) When the base and height are known, area can be calculated from Area = ½ (base of triangle x height of triangle)
(3) When two sides and the angle between them are given, it can be calculated using Area = ½absinC, where C is the angle contained between side lengths a and c

The area of any straight sided figure can be calculated by splitting it into triangles and summing the individual areas.
Given that the shape of a land in figure 3.1a, the area can be determined by splitting it into three triangles as shown in figure 3.1b and summing

(a)



(b)

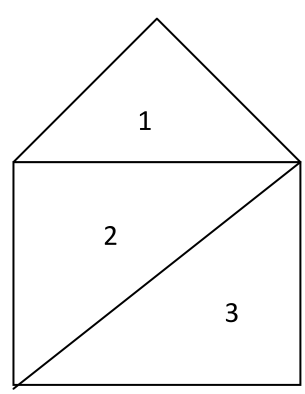


Figure 3.1

3.1.2 Area from coordinates

Areas of plots bounded by straight line can be calculated from coordinates of the junctions of their sides. This is achieved using the cross coordinate method.

Consider a triangle ABC with coordinates of the vertices A, B and C as (N_1, E_1), (N_2, E_2) and (N_3, E_3), respectively.

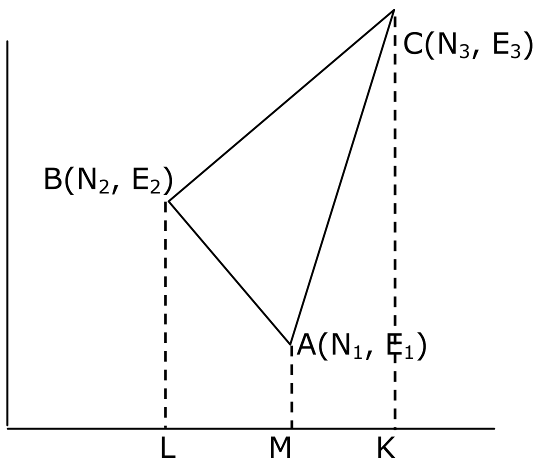


Figure 3.2 Triangular piece of land

Area of trapezium = (mean height x width)
The area of ABC = BCKL - BAML – ACKM

The area of 2ABC = 2BCKL -2BAML – 2ACKM

2 x Area of BCKL = $(N_3 + N_1)(E_3 - E_2)$

2 x Area of BAML = $(N_3 + N_1)(E_3 - E_1)$

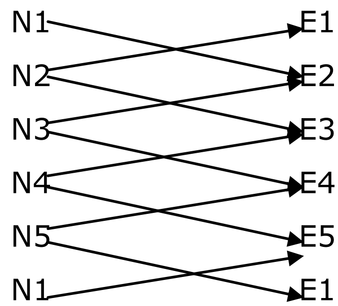
2 x Area of ACKM = $(N_1 + N_2)(E_1 - E_2)$

2 x Area of ABC = $(N_1E_2 + N_2E_3 + N_3E_1) - (N_1E_3 + N_2E_1 + N_3E_2)$

Similarly for every succeeding triangle that may be formed as a result of an additional side and the area of the whole polygon will be given. For a polygon with five vertices, the area is as follows:

2 x Area= $(N_1E_2 + N_2E_3 + N_3E_4 + N_4E_5 + N_5E_1) - (N_1E_3 + N_2E_1 + N_3E_2 + N_4E_3 + N_5E_4)$

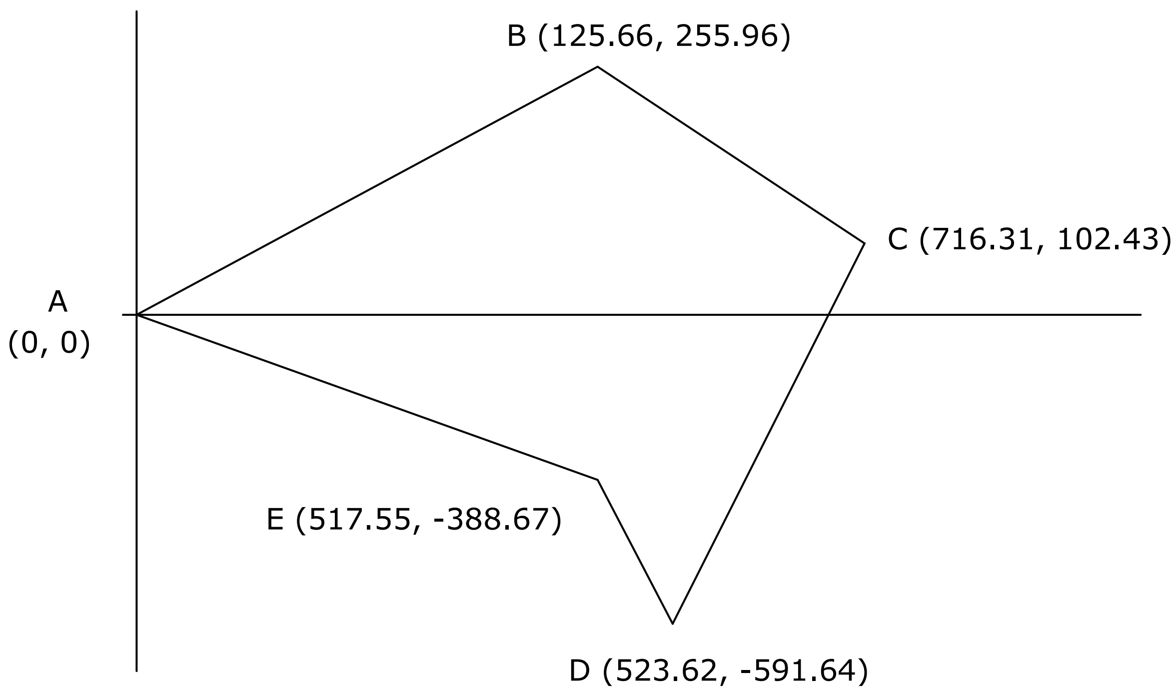
In tabular form, the above polygon can be writes as



The continuous lines are multiplied and the products added. The dotted lines are also multiplied and the products added. The difference between these two is twice the area i.e. 2A. Note that only A is required.

EXAMPLE

Calculate the area of plot in the figure below. Use the given coordinates.



$A(x_1, y_1) = (0, 0)$

$B(x_2, y_2) = (125.66, 255.96)$

$C(x_3, y_3) = (716.31, 102.43)$

$D(x_4, y_4) = (523.62, -591.64)$

$E(x_5, y_5) = (517.55, -388.67)$

0	0
255.96	125.66
102.43	716.31
-591.64	523.62
-388.67	517.55
0	0

2 x Area = (0)(125.66) + (255.96)(716.31) + (102.43)(523.62) + (-591.64)(517.55) + (-388.67)(0) – (255.96)(0) + (102.43)(125.66) + (-591.64)(716.31) + (-388.67)(523.62) + (0)(517.55)

= (0 + 183346.71 + 53634.40 – 306203.28 – 0) – (0 + 12871.35 – 423797.65 – 203515.39 + 0)

= (-69222.17) – (-614441.69)

= 545219.52

Area = 545219.52 /2 = 272609.76 sq ft

3.2 Areas enclosed by irregular lines

For such cases only approximate results can be achieved. However, methods are adopted which give the best approximations.

3.2.1 Trapezoidal rule

Trapezoidal rule assumes that if the interval between the offsets is small, the boundary between the offsets can be approximated to a straight line.

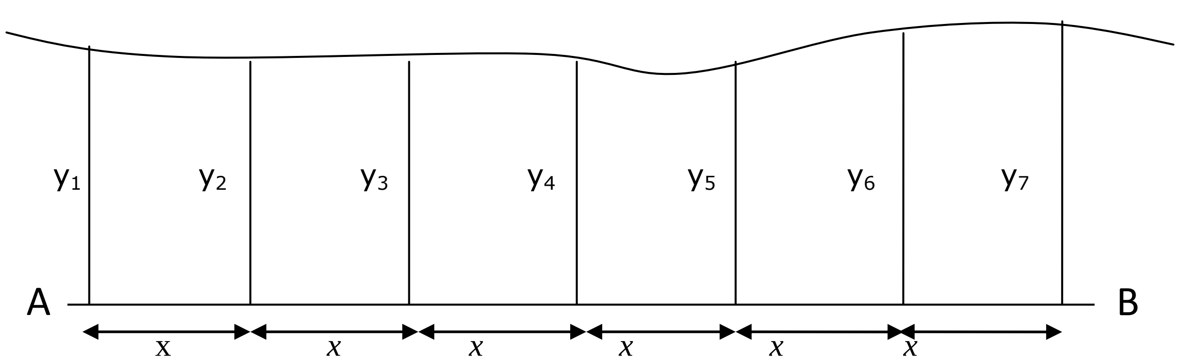


Figure 3.3

Divide inter (a, b) into series of trapezoids.

X is the common distance between the ordinates

y_1, y_2, … , y_n are the scaled ordinate lengths

From this concept, the areas of trapezoids are determined.

Ai is the area of i (i = 1 to 6)

Area of the first trapezoid, A_1 = x $\left(\frac{y1+y2}{2}\right)$

Area of the 2nd trapezoid, A_2 = x $\left(\frac{y2+y3}{2}\right)$

.

.

.

Area of the 6th trapezoid, A_6 = x $\left(\frac{y6+y7}{2}\right)$

Summing up the Ai's

Total Area = $\frac{x}{2}$ (y1 + 2y2 + 2y3 + 2y4 + 2y5 + 2y6 + y7)

= x $\left(\frac{y1+y7}{2} + y2 + y3 + y4 + y5 + y6\right)$

Example

The following offsets, 8m apart were measured at right angles from a traverse line to an irregular boundary 0m, 2.3m, 5.5m, 7.9m, 8.6m, 6.9m, 7.3m, 6.2m, 3.1m, and 0m.

Calculate the area between the traverse line and the irregular boundary.

Solution

Area = $\frac{8.0}{2}$ (0 + 0 + 2(2.4 + 5.5 + 7.9 + 8.6 + 7.3 + 6.2 + 3.1))

= 4 (2) (47.8)

` = 382.4 m^2

3.2.2 Simpson's rule

Simpson's rule assumes that instead of the boundary line being made up of series of straight lines, it consists of a series of parabolic arcs.

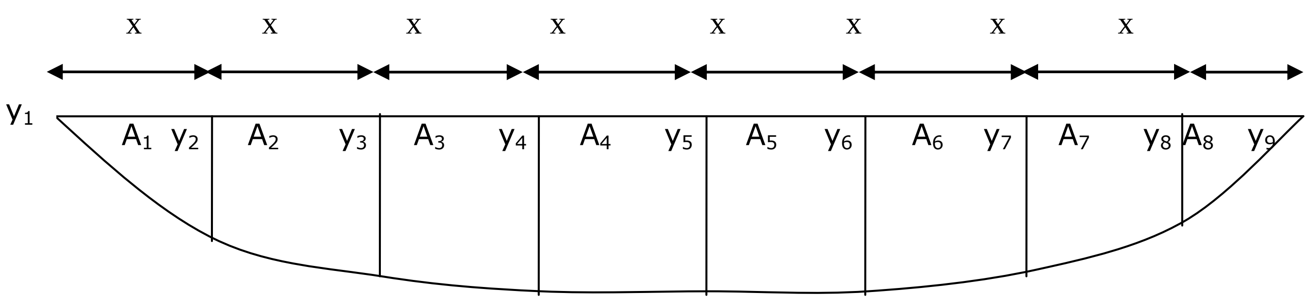


Figure 3.4

The baseline must be divided into even number of equal intervals. This will give an odd number of heights or ordinates as shown in Figure 3.4.

Simpson's rule considers offsets in sets of three and it can be shown that the area between offsets 1 and 3 is:

$A1 + A2 = {}^{x}/_{3}(y1 + 4y2 + y3)$

Similarly area between offsets 3 and 4 is:

$A3 + A4 = {}^{x}/_{3}(y3 + 4y4 + y5)$

Hence, in general

$\text{Total area} = {}^{x}/_{3}(y1 + y9 + y7) + 4(y2 + y4 + y6 + y8)$

An odd number of coordinates is essential in simpson's rule. If an even number of ordinates is given, neglect the first one, calculate by simpson's rule and then calculate the neglected by trapezoidal rule and add.

EXAMPLE

A strip of land is 1920 meters long. This length is marked off into eight equal intervals and the consecutive breadths were measured at the ends of the intervals as follows: 10, 26, 30, 36, 40, 48, 24, 12 and 10 meters

Apply Simpson's rule and calculate the area in hectares

Solution

Each interval = 1920/8 = 240m

Sum of the first and last ordinates = 10 + 10 = 20m

Twice sum of all other odd ordinates = 2(30+40+24) = 188m

Four times sum of all other even ordinates = 4(26 + 36 + 48 + 12)

Area = 240/3 (20+188+488)

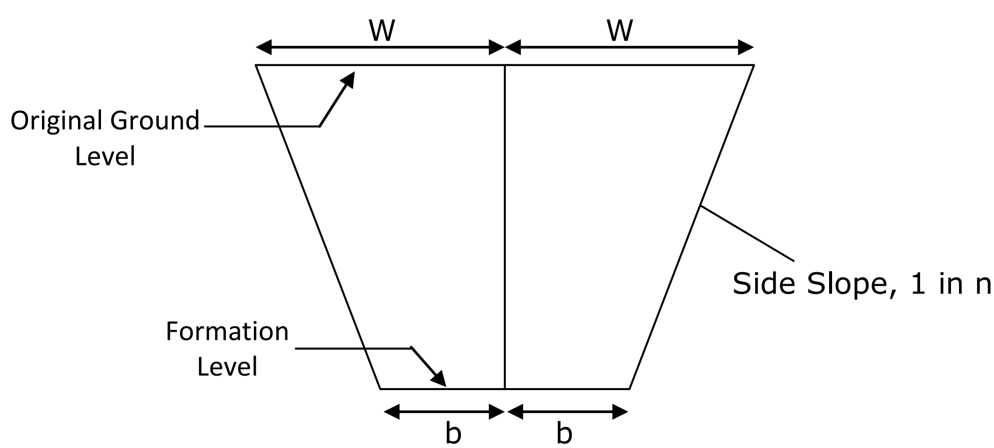
= 55680m^2

=5.57 ha

3.3 Calculation of cross sectional areas

In the construction of a road, railway, large diameter underground pipeline or similar, having set out the proposed center line on the ground, levels are taken at regular intervals both along it and at right angles to it, to obtain the longitudinal and cross sections. Each cross section (CS) is drawn and the area between the existing and proposed levels is calculated.

3.3.1 Existing ground level (Horizontal)



The figure 3.5 shows a section drawing with cutting formed in an area where the existing ground level is constant.

Depth at center = h

Side Slopes = 1 in n

Formation width = 2b

Side width = W

Plan width = 2W

From Figure

Area of cross section = h(2b + nh)

Plan width = 2W = 2(b+nh)

3.3.2 Cutting or embankment on sloping ground (the two level section)

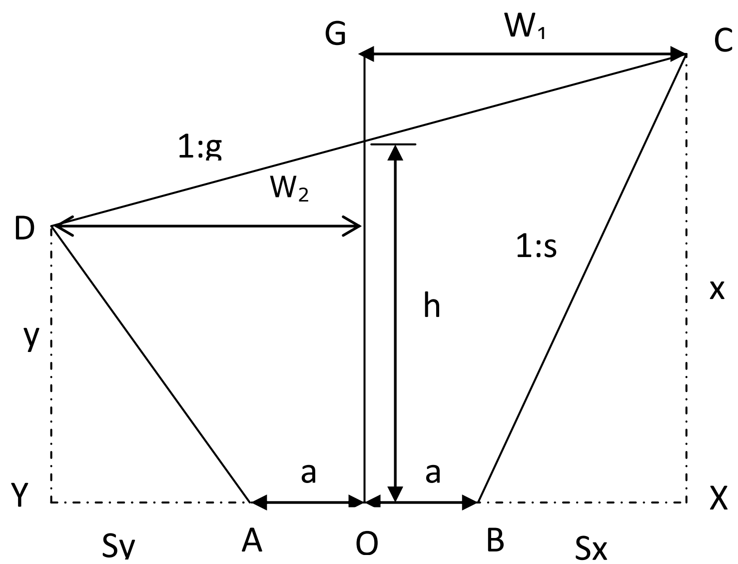


Figure 3.6

To find the area of cross-section ABCD and half widths, given side slope, ground slope formation width and height at center in figure 3.6.

Let the formation width, AB = 2a

Height at center = h

Side slope = 1:s

Ground slope = 1:g

To find half widths CG (W_1) and DH (w_2)

Erect perpendicular CX and DY on line AB produced

Then

$$CX = GO = \frac{BX}{s} = \frac{W_1 - a}{s}$$

Therefore, GM = GO − h

But GM = $\frac{W_1}{g}$

$$= \frac{W_1}{g} = \frac{W_1 - a}{s} - h = \frac{W_1 - a - hs}{s}$$

Cross multiplying

$W_1s = W_1g - ag - ghs$

$W_1(g - s) = ag + ghs$

$W_1 = \frac{g(a+hs)}{(g-s)}$

Similarly,

DY = MO-MH

DY = h − MH

MH = h − DY

Also, MH = $\frac{W_2}{g}$

DY = $\frac{YA}{s} = \frac{W_2 - a}{s}$

MH = h - $\frac{W_2-a}{S} = \frac{hs - W_2 + a}{s}$

$\frac{W_2}{g} = \frac{hs - W_2 + a}{s}$

$W_2s = ghs - W_2g + ag$

$W_2(g+s) = g(a+hs)$

$W_2 = \frac{g\,(a+hs)}{g+s)}$

To find area ABCD

Section ABCD = Section CXYD − Section CXB − Section DYA

Area of section ABCD = Area of Section CXYD − Area of section CXB − Area of section DYA

$$= XY\left(\frac{CX + DY}{2}\right) - ½(CX)(BX) - ½(DY)(AY)$$

CX = BX = $\frac{W_1 - a}{s}$ XY = $W_1 + W_2$

Area of section ABCD

$$= \left(\frac{W1+W2}{2}\right)\left(\frac{W1-a}{s} + \frac{W2-a}{s}\right) - ½\left(\frac{W1-a}{s}\right)(W_1 - a) - ½\left(\frac{W2-a}{s}\right)(W_2 - a)$$

$$= \left(\frac{W1-W2}{2}\right)\left(\frac{W1-a+W2-a}{s}\right) = ½\left(\frac{W1-a}{s}\right) = ½\left(\frac{W2-a}{s}\right)$$

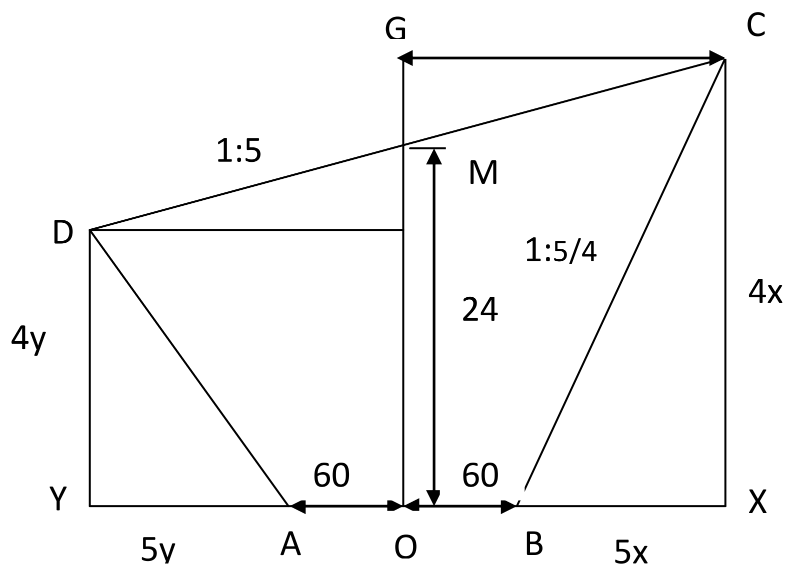
$$= \frac{1}{2s}\left\{(W_1+W_2)(W_1+W_2-2a) - (W_1 - a)^2 - (W_2 - a)^2\right\}$$

$$= \frac{1}{2s}(2W_1W_2-2a^2) = \frac{1}{s}(W_1W_2-a^2)$$

Area of ABCD $= \frac{1}{s}(W_1W_2-a^2)$

Note that a in the formula is half the formation width.

EXAMPLE



Given
Formation width = 120m
Height at centre = 24
Side slope = 1:11/4
Ground slope = 1:5

Calculate (i) the half width
(ii) the plan width
(iii) area of section ABCD

Solution

$W_1 = g\left(\frac{a + hs}{g - s}\right) = 5\left(\frac{60 + 30}{5 - 5/4}\right) = 120m$

$W_2 = g\left(\frac{a + hs}{g + s}\right) = 5\left(\frac{60 + 30}{5 + 5/4}\right) = 72m$

(i) Plan width = $W_1 + W_2$

= 120 + 72

= 192m

(ii) Area of section ABCD $= \frac{1}{s}(W_1W_2 - a^2)$

= 4/5 (120(72)) – 60)

= 4/5 (8640 – 3600)

= $4032m^2$

From first principle,

GC = 5GM

GC = 60 + 5x

CM = 4x – 24

60 + 5x = 5(4x – 24)

15x = 180

x = 12, CX = 48, BX = 6

Also, DH = 5HM

HM = 24 -4y

60 +5y = 5(24-4y)

y = 2.4

D = 5(24-4y)

y = 2.4

DY =9.6, AY = 12

W_1 = GC = 60 + 5x

$= 60 + 5(12)$

$= 120m$

$W_2 = 60 + 5x$

$= 60 + 5(2.4)$

$= 72m$

Area of Section ABCD = Area section CXYD – Area Section CXB – Area Section DYA

$$= XY\left(\frac{CX + DY}{2}\right) - ½(CX)(BX) - ½(DY)(AY)$$

$$= (120+60+12)\left(\frac{48+9.6}{2}\right) - \left(\frac{9.6 \times 12}{2}\right) - \left(\frac{48 \times 60}{2}\right)$$

$= 4032m^2$

Plan width = 120 + 72 =192m